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EE 270 - Applied Quantum Mechanics

Hermitian Operators

(1) Complex conjugate, A^*, formed by taking the complex conjugate ($i \rightarrow -i$) of each element, where $i = \sqrt{-1}$
(2) Adjoint, $A^\dagger$, formed by transposing A^*,

$$A^\dagger = (A^T)^* = (A^*)^T$$

(3) Hermitian matrix : The matrix A is labeled Hermitian (or self-adjoint) if,

$$A^\dagger = A$$

If A is real, then it comes $A^\dagger = A^T$, and matrices are real symmetric matrices. In quantum mechanics, matrices are usually constructed to be Hermitian.

Example :

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad A^* = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad (A^T)^* = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

We conclude that our matrix is Hermitian (self-adjoint) since $A^\dagger = (A^T)^* = (A^*)^T = A$.

(4) If A is Hermitian, its eigenvalues are real and the eigenvectors are orthogonal. Let λ_i and λ_j be two eigenvalues and $|r_i\rangle$ and $\left|r_j\right\rangle$ the corresponding eigenvectors of A, a Hermitian matrix. Then,

$$A\,|r_i\rangle = \lambda_i\,|r_i\rangle$$

$$A\left|r_j\right\rangle = \lambda_j\left|r_j\right\rangle$$

and,

$$\left\langle r_j\middle|A\middle|r_i\right\rangle = \lambda_i\left\langle r_j\middle|r_i\right\rangle$$

$$\left\langle r_i \middle| A \middle| r_j \right\rangle = \lambda_j \left\langle r_i \middle| r_j \right\rangle$$

Taking the adjoint of $\left\langle r_j \middle| A \middle| r_i \right\rangle$, we have,

$$\left\langle r_i \middle| A^\dagger \middle| r_j \right\rangle = \lambda_i^* \left\langle r_i \middle| r_j \right\rangle$$

As the operator is Hermitian, we get,

$$\left\langle r_i \middle| A \middle| r_j \right\rangle = \lambda_i^* \left\langle r_i \middle| r_j \right\rangle$$

Subtracting the aforementioned equations, we get,

$$(\lambda_j - \lambda_i^*) \left\langle r_i \middle| r_j \right\rangle = 0$$

This is a general result for all possible combinations of i and j. First, let $j = i$, we get,

$$(\lambda_i - \lambda_i^*) \langle r_i | r_i \rangle = 0$$

Since $\langle r_i | r_i \rangle = 0$ would be a trivial solution, we conclude that

$$\lambda_i = \lambda_j^*$$

or λ_i is real for all i.
Second, for $i \neq j$, and $\lambda_i \neq \lambda_j$,

$$(\lambda_i - \lambda_j) \left\langle r_i \middle| r_j \right\rangle = 0$$

or

$$\left\langle r_i \middle| r_j \right\rangle = 0$$

which means that the eigenvectors of distinct eigenvalues are orthogonal. this last equation is a generalization of orthogonality in the complex space.
Note : If $\lambda_i = \lambda_j$ (degenerate case), $|r_i\rangle$ is not automatically orthogonal to $\left| r_j \right\rangle$ but it may be made orthogonal.

(5) The operator A is Hermitian if and only if,

$$\int \psi_1^* A \psi_2 d\tau = \int (A\psi_1)^* \psi_2 d\tau$$

The adjoint $A^\dagger$ of an operator A is defined by,

$$\int \psi_1^* A^\dagger \psi_2 d\tau = \int (A\psi_1)^* \psi_2 d\tau$$

The expectation value $< a >$ of an operator A is defined as,

$$< a >= \int \psi^* A \psi d\tau$$

In the framework of quantum mechanics, <a> corresponds to the result of a measurement of the physical quantity represented by A when the physical system is in a state described by the wave function ψ. If we require A to be Hermitian, it is easy to show that <a> is real (as would expect from the measurement).
Taking the complex conjugate of the above equation,

$$< a >^* = \left[\int \psi^* A \psi d\tau \right]^*$$

$$< a >^* = \int \psi A^* \psi^* d\tau$$

Rearranging the factors in the integral, we have

$$< a >^* = \int (A\psi)^* \psi d\tau$$

Then applying the definition of Hermitian operator, we get,

$$< a >^* = \int \psi^* A \psi d\tau = < a >$$